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1 Brief Notes on Kinematics and 4-Momentum

The aim of this note is to introduce basic relativistic kinematics and natural units in particle physics. It is organised as follows: a brief introduction, an aside on natural units, and then a discussion of the basic equation and finally some comments on the applicability of the expression to systems of particles rather than a single particle. One of many reasons to appreciate some relativistic kinematics is that particle production takes place in the "centre-of-mass" frame of the particles in question, which is often difference from the laboratory frame. Understanding kinematics allows you to relate one system to the other using simple algebra, without explicitly performing any Lorentz transformations.

2 Introduction to Relativistic Kinematics

The relativistic relationship between the mass, m, 3-momentum, $\vec{p}$ and total energy, E, of a particle may be written as

$$E^2 = p^2c^2 + m^2c^4, \tag{1}$$

where $p = |\vec{p}|$ and c is the speed of light.

All three parts of this expression have dimensions of energy, such as GeV, and so it is convenient to measure momentum in units such as GeV/c and mass in units of GeV/c^2. In Eq. 1, this gives the dimensions:

$$[E(\text{GeV})] = [\vec{p}(\text{GeV}/c)]^2c^2 + [m(\text{GeV}/c^2)]^2c^4 \tag{2}$$

As c cancels out, it is common practise for particle physicists to work in a system of so-called "natural units", in which we set both Plank's constant and the speed of light to unity, i.e. $\hbar = c = 1$. Using this convention, Eq. 1 simplifies to

$$E^2 = p^2 + m^2. \tag{3}$$

You will encounter in the literature mass (momentum) quoted in both GeV/c^2 (GeV/c) and GeV and should not be concerned by this.

2.1 Aside on Systems of Units

The charge of observed particles is quantised in multiples of the charge on the electron. It is therefore natural consider charge in units of the electron charge, rather than in multiples of $1.602176462 \times 10^{-19}$ C. Similarly, a simple choice for a unit of energy is the electron-Volt, where 1 eV is the energy gained by accelerating an electron through a potential difference of 1 V, i.e. 1.6×10^{-19} J. Energies which are appropriate to most experiments are MeV (10^6 eV), GeV (10^9 eV), TeV (10^{12} eV).

While the SI system of units has many merits, it is not usually appropriate for high energy physics, e.g. the mass of the electron is usually quoted as 0.511 MeV/c^2 rather than 9.1×10^{-31} kg. As long as we state clearly which system of units is in use, we will be able to convert to SI should you ever need to do so. (We could, for example, choose to work in a system in which we define c to be 3×10^8 ms^{-1}, 1 light year per year, 1 foot per ns, etc., and still be be able to perform calculation, but are stronly advised against doing so.)

A very useful conversion constant is the combination $\hbar c$. This is given by

$$\begin{aligned}
\hbar c &= 1.05 \times 10^{-34}\ (\text{Js}) \times 3 \times 10^8\ (\text{ms}^{-1}) \quad (\text{SI units}) \\
&= 3.15 \times 10^{-26}\ (\text{Jm}) \\
&= \frac{3.15 \times 10^{-26}\ (\text{Jm})}{1.6 \times 10^{-19}\ (\text{J/eV})}\ \text{eV m} \\
&= 1.97 \times 10^{-7}\ \text{eV m} = 197\ \text{MeV fm} \qquad (4)
\end{aligned}$$

We can see that this has the correct dimensions if we recall the uncertainty relationship

$$\begin{aligned} \hbar &= \Delta E \Delta t = [\text{energy}][\text{time}] \\ \hbar c &= [\text{energy}][\text{time}][\text{distance/time}] \\ &= [\text{energy}][\text{distance}] \end{aligned} \tag{5}$$

We can illustrate this system of units by taking an example the example of the range of the weak force mediated by a massive W boson which exists for a short time, given by the uncertainty principle. As the minimum energy that the W can have is its mass (80 GeV), and as it must travel at less than the speed of light, c, the maximum distance it can travel is approximately given:

$$\begin{aligned} \text{range} &\sim c \times \Delta t = c \times \frac{\hbar}{\Delta E} \qquad (6) \\ &= \frac{197 \text{ (MeV fm)}}{80 \text{ (GeV)}} \simeq 0.0025 \text{ fm.} \end{aligned}$$

2.2 Discussion

We next look at two aspects of Eq. 1.

Firstly, we consider the non-relativistic behaviour, i.e. for particle speeds, $v \ll c$, to compare with our knowledge of classical mechanics.

$$E^2 = p^2 + m^2 \tag{7}$$

Rearranging, gives

$$E = m(1 + p^2/m^2)^{\frac{1}{2}} \tag{8}$$

$$= m(1 + \frac{1}{2}p^2/m^2 + \ldots) \tag{9}$$

For low speeds (and only for low speeds), p/m is small, therefore we can approximate and retain only the first term in p/m, giving

$$E \simeq m + \frac{1}{2}mv^2 \tag{10}$$

If you prefer to include c's, this is $E = mc^2$ plus kinetic energy, as we would expect.

Secondly, we consider the relativistic Lorentz factor, which is used to quantify time dilation and related effects between two frames of reference in relative motion at a speed of $\beta \equiv v/c$. The Lorentz factor is defined as

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{1-\beta^2}} \qquad (11) \\ \gamma^2(1-\beta^2) &= 1 \\ \gamma^2 &= \gamma^2\beta^2 + 1 \end{aligned}$$

Multiply throughout by m^2 gives

$$\gamma^2 m^2 = \gamma^2 m^2 \beta^2 + m^2 \tag{12}$$

Comparing the forms of equations 7 and 12, we can infer

$$E = \gamma m \qquad (\text{or} = \gamma m c^2) \tag{13}$$

$$p = \gamma m \beta \qquad (\text{or} = \gamma m \beta c = \gamma m v). \tag{14}$$

The forms in parenthesis are given as the importance of the Lorentz gamma factor may be more transparent when seen using the more familiar expressions for energy and momentum.

3 4-Momentum

We are familiar with vectors having three components, such as Cartesian co-ordinates, or momentum. We know the rules by which such vectors may be manipulated, such as addition, modulus, rotation. An important feature of a vector is that it may be rotated in co-ordinate space without altering its length, that is to say its length is *invariant under rotations*.

A similar statement can be made about the mass of a particle. We know that energy and momentum of a particle are initimately related and can introduce a quantity called the *4-momentum*, P, of a particle of mass, m, energy, E, and (vector) momentum, $\vec{p}$, as

$$P = (E, \vec{p}), \tag{15}$$

where the modulus (length) of the 4-momentum is *defined* through

$$P^2 = E^2 - p^2 \qquad (\equiv m^2) \tag{16}$$

The minus sign is extremely important in this definition.[1] For a single particle, it is clear that this definition gives a quantity which is constant, namely its mass, and so will be the same regardless of the size of E or p. An equivalent way of saying the same thing is that the scalar product of two 4-momenta (of which P^2 is just a special case, scalar product with itself) is *invariant under Lorentz transformations*.

In the same way that the magnitude of a 3-vector is invariant under a rotation in co-ordinate space, the magnitude of a 4-momentum is invariant under Lorentz transformations. Extending the analogy, if several 3-vectors are added together, the resultant vector is likewise invariant under rotations. Similarly, if several 4-momenta are added together, the magnitude of the resultant 4-momentum is also invariant. This quantity is particularly important and is referrred to as the *invariant mass* of a system. If a particle decays to several other particles, you can add their 4-momenta together to calculate directly the mass of the parent particle, i.e.

$$P^2 = (\sum E)^2 - (\sum p_x)^2 - (\sum p_y)^2 - (\sum p_z)^2, \tag{17}$$

where the summation extends over all particles of a decay. Again, if all particles in the initial state or final state are considered, this gives the centre-of-mass energy of the reaction.

One example of the use of this is in considering the minimum energy required to form a new particle by colliding a particle with a fixed target. This is done by considering the minimum energy that the new particle can have in its own rest frame, and the energy of the incident and target particles in the lab. frame, then calculating and equating the invarariant mass in the two systems.

[1] Formally, there is more to the algebra and we should really introduce covariant and contravriant indices, P^μ, etc, see e.g. Halzen and Martin, Sec. 3.2.. Some authors introduce factors of the imaginary i in the 4-momentum definition, others use metric tensors, you will see both in the literature.