Year 2 Particles and Nuclei – Lectures 9 & 10

Nuclear Masses and Stability

A nucleus is a bound state of protons and neutrons. Its composition may be characterised by the number of protons (Z) and the total number of protons+neutrons (A). How strongly it is bound can be expressed in terms of the *mass deficit*:

$$\Delta M(Z, A) = M(Z, A) - Z(M_p + m_e) + NM_n$$

where M(Z,A) is the mass of the atom, and M_p, M_n and m_e are the masses of the proton, neutron and electron respectively (the electrons are included because it is generally the mass of the neutral atom which is measured), and N = A – Z is the number of neutrons. For a bound state this quantity must be negative.

When multiplied by $-c^2$ this difference gives us the net *binding energy* of the nucleus, $B = -\Delta Mc^2$. When discussing stability, a particularly useful quantity is the *binding energy per nucleon*, B/A. This is the average binding energy of the nucleons in a particular nucleus, which is not necessarily the energy needed to remove an individual nucleon (proton or neutron) from the nucleus.

Whether a nucleus of a given Z and A is stable or not depends on whether it can split into 2 or more nuclei with a larger total binding energy. If it can, the final state products have a lower total mass than the initial nucleus, and so the decay is energetically possible (which generally means that sooner or later it will happen). However, the nucleus is a complex object: many particles interacting via a strong, relatively poorly-modelled short-range force. So whereas it's possible to predict the energy levels of electrons in atoms very precisely, for the nucleus we do not have a similar ability. But if we are to understand nuclear stability, nuclear reactions and decays, we need some way of describing this. One way this is done is via the *semi-empirical mass formula*.

Liquid drop model and the semi-empirical mass formula

Although there are exceptions, most nuclei can be fairly well described as homogeneous spheres of matter, with a radius $R \approx 1.21\mathrm{A}^{1/3}$ fm. This means the volume is simply proportional to A, and hence implies an approximately equal density for all nuclei. This is known as the "liquid drop" model, since it is what we would expect for droplets of a homogeneous fluid.

This forms the theoretical basis for the semi-empirical mass formula. The formula is "semi-empirical" because while it contains many constants which cannot be predicted and need to be measured, it is inspired by a theoretical model of nuclear structure. It parameterises the mass of the atom as the sum of 6 terms:

$$\mathrm{M(Z,A)} = \sum_{i=0}^{5} f_i(\mathrm{Z,A})$$

where:

- f_0(Z, A) = Z(M_p + m_e) + (A-Z)M_n is the mass of the constituents of the atom.
- f_1(Z, A) = - a_1A is the "volume correction", which accounts for the binding energy due to the nuclear force.

  The idea is that since the nuclear force has a short range, each nucleon only interacts with its immediate neighbours. As all nuclei have (in this model) the same density, the number of neighbours is the same in all nuclei, and so the net binding energy is just proportional to the number of nucleons (A).

- f_2(Z, A) = a_2A$^{2/3}$ is the "surface correction".

  Nucleons near the surface of the nucleus have fewer neighbours than those deeper in, so the volume term over-estimates the binding energy of these. Since the number of surface nucleons is proportional to the surface area, if the volume is proportional to A then this should be proportional to A$^{2/3}$.

- $f_3(\mathrm{Z,A}) = a_3 \dfrac{\mathrm{Z(Z-1)}}{\mathrm{A}^{1/3}} \approx a_3 \dfrac{\mathrm{Z}^2}{\mathrm{A}^{1/3}}$ is the "Coulomb correction".

  This corrects for the mutual repulsion of the protons. As an EM potential it is proportional to 1/R, i.e. to A$^{-1/3}$, and each of the Z protons is being repelled by its (Z-1) companions. For heavy nuclei the approximation of using Z^2 instead of Z(Z-1) is adequate.

- $f_4(\mathrm{Z,A}) = a_4 \dfrac{(\mathrm{Z} - \mathrm{A}/2)^2}{\mathrm{A}}$ is the "asymmetry correction".

  Imagine that there is a set of discrete energy states for nucleons bound in the nucleus (in QM this is true for any bound particle, though how those states are spaced depends on how the particle is confined). Due to the Pauli principle we cannot have 2 identical particles (protons or neutrons) in the same state. So if a nucleus has Z = A/2, the Z lowest-energy states will be occupied by both protons and neutrons. If we keep the same A replace some of the protons with neutrons (so Z < A/2), the extra neutrons will have to occupy higher-energy states. This is what this term describes.

- $$f_5(\mathrm{Z,A}) = \left.\begin{array}{ll} -f(\mathrm{A}) & : \mathrm{Z} = \text{even and } \mathrm{N} = \text{even} \\ = 0 & : \mathrm{Z} \text{ even} + \mathrm{N} \text{ odd or } \mathrm{Z} \text{ odd} + \mathrm{N} \text{ even} \\ = +f(\mathrm{A}) & : \mathrm{Z} = \text{odd and } \mathrm{N} = \text{odd} \end{array}\right\} \text{ is the "pairing correction"}$$

  $f(\mathrm{A}) = a_5 \mathrm{A}^{-1/2}$ is a common empirical parameterisation of this term

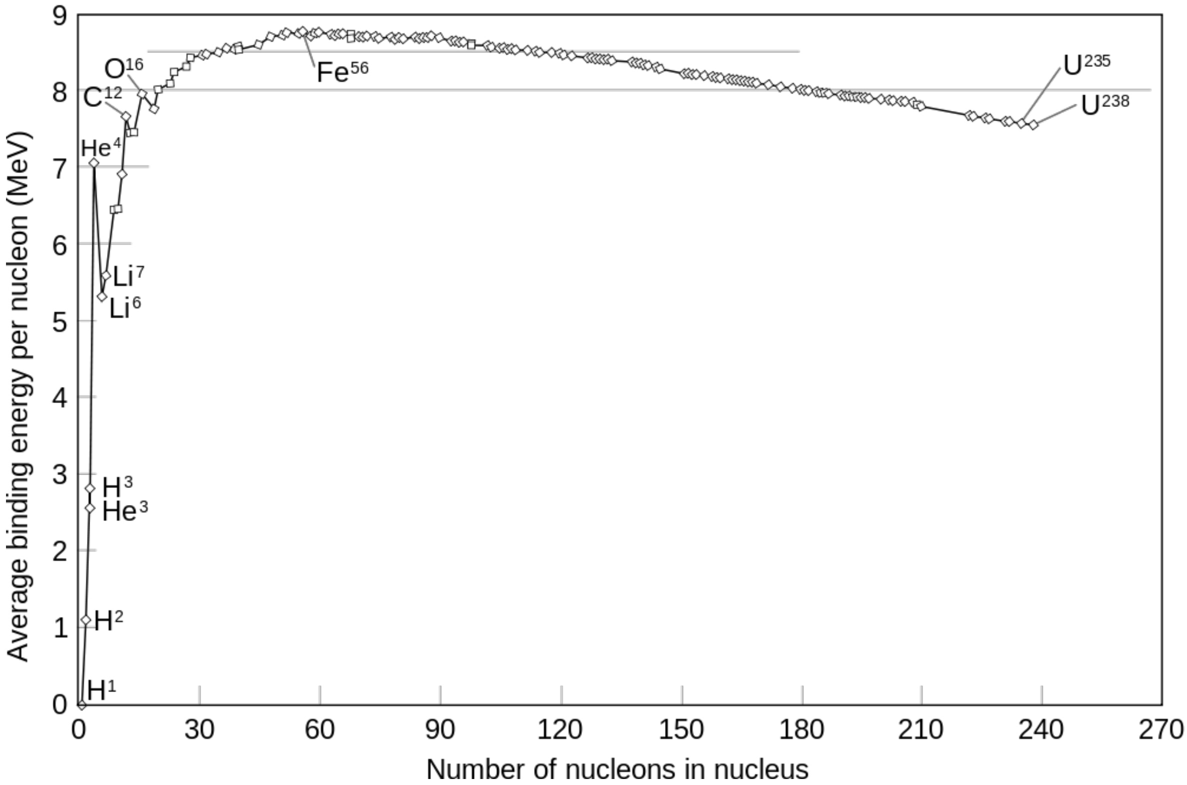
  It is observed that nuclei with even numbers of protons or neutrons are more stable. This is understood to be due to nucleons “pairing up”, with 2 protons or 2 neutrons in the same state but with opposite spins. These pairs have a greater overlap of their wavefunctions in space so feel a stronger than average mutual attraction. This term then reduces the estimated binding energy for nuclei where both protons and neutrons are all paired (“even-even” nuclei), and increases it where there is one unpaired nucleon of each type (“odd-odd”).

Values of these 5 parameters (a_i) are obtained from fits to the measured masses of nuclei. One set of values is a_1 = 15.56, a_2 = 17.23, a_3 = 0.697, a_4 = 93.14, a_5 = 12 (all MeV/c^2).

The parameters are sometimes referred to as a_V, a_S, a_C, a_A and a_P, for “Volume”, “Surface”, “Coulomb”, “Asymmetry” and “Pairing” respectively.

This formula does not give a good description for light nuclei (few nucleons, liquid drop not a good approximation), but for A > 20 it works pretty well.

If we look at the average binding energy per nucleon, we can understand its main features in terms of the SEMF.



From the definition above, the binding energy equals minus the sum of the 5 "correction" terms × c^2.

$$B = -\sum_{i=1}^{5} a_i f_i \left(\mathrm{Z}, \mathrm{A} \right)$$

$$\frac{B}{\mathrm{A}} = a_1 - a_2 A^{1/3} - a_3 \frac{\mathrm{Z}^2}{\mathrm{A}^{4/3}} - a_4 \frac{\left(\mathrm{Z} - \mathrm{A}/2 \right)^2}{\mathrm{A}^2} - \frac{f_5(\mathrm{Z},\mathrm{A})}{\mathrm{A}}$$

For light nuclei, the volume term dominates, i.e. the trend is for the binding energy/nucleon to increase as the number of nucleons does, because the volume increases faster than the surface area. However, as Z increases, the Coulomb correction becomes more significant. This can be offset by increasing the number of neutrons (increasing the volume term without increasing the Coulomb one), but that requires neutrons to be added to higher-energy, less tightly-bound states (described by the asymmetry correction). This slows the rise in average binding energy, and for nuclei above iron the average binding energy per nucleon in stable nuclei gradually decreases. Eventually it decreases to the point where no stable nucleus of a given Z exists: even with the number of neutrons which maximises the binding energy/nucleon for that element (i.e. the most stable isotope of that element) it is energetically favourable for the nucleus to disintegrate.

Nuclear Decays

All nuclear decays (and indeed particle decays discussed earlier) follow the radioactive decay law:

$$N\left(t \right) = N_0 e^{-t/\tau}$$

where τ is the decay constant, which is also equal to the mean lifetime of the state. This is related to the half life, the time for half of the initial nuclei to decay, by $t_{1/2} = \tau \ln\left(2 \right)$.

This decay law follows from the fact that the probability for a radioactive nucleus to decay per unit time is a constant, independent of how long since it was created in that state.

Beta decay

In beta decay a nucleon changes type, emitting an electron and a neutrino. Such decays change the Z of the nucleus but not A.

There are two types of beta decay:

- $\beta-: \; n \rightarrow p \; e^- \; \bar{\nu}_e$ Z → Z + 1
- $\beta+: \; p \rightarrow n \; e^+ \; \nu_e$ Z → Z - 1

It is the mass difference between nuclei of the same A but different Z which determines whether these reactions are possible. From the above reactions we can see that the condition is that the mass of the decaying nucleus must exceed that of the one it is decaying to by an amount > the mass of the electron (since the mass of the neutrino is negligible). When expressed in terms of atomic masses, this becomes:

- $\beta-$: M(Z, A) > M(Z+1, A)

since the mass of the atom with Z+1 includes the extra electron already, and

- $\beta+$: M(Z, A) > M(Z–1, A) + $2m_e$

since M(Z–1, A) includes 1 fewer electrons than M(Z, A) , in addition to which the mass of the positron must be added.

There is also the related process of electron capture:

- $EC: \; p + e \rightarrow n + \nu_e$ Z → Z - 1

Here the energy barrier is lower, since we do not have to create an electron. Due to the short-range nature of the weak force it is usually only electrons from the innermost shell which are captured (K shell in spectrographic notation), which in high-Z atoms can have a significant binding energy. Thus in terms of atomic masses, the condition for this reaction to be possible is:

- EC : M(Z, A) > M(Z–1, A) + ε

where ε is the excitation energy of the atomic shell of the captured electron in the daughter nucleus. This is because although the daughter atom already has the correct number of electrons, one will be in a highly excited shell and there will be a vacancy in the innermost shell, so the energy of the atom will be higher than its ground state. Electron capture decays are followed by a sequence of photon emissions as the electrons de-excite.

We can use the SEMF to predict stability against beta decay. To do this, let's look at how the masses of nuclei with the same A (isobars) vary with Z:

If we group terms in the formula as powers of Z, we obtain:

$$M(\mathrm{Z,A}) = \alpha \mathrm{A} - \beta \mathrm{Z} + \gamma \mathrm{Z}^2 + \frac{\delta}{\sqrt{\mathrm{A}}}$$

where:

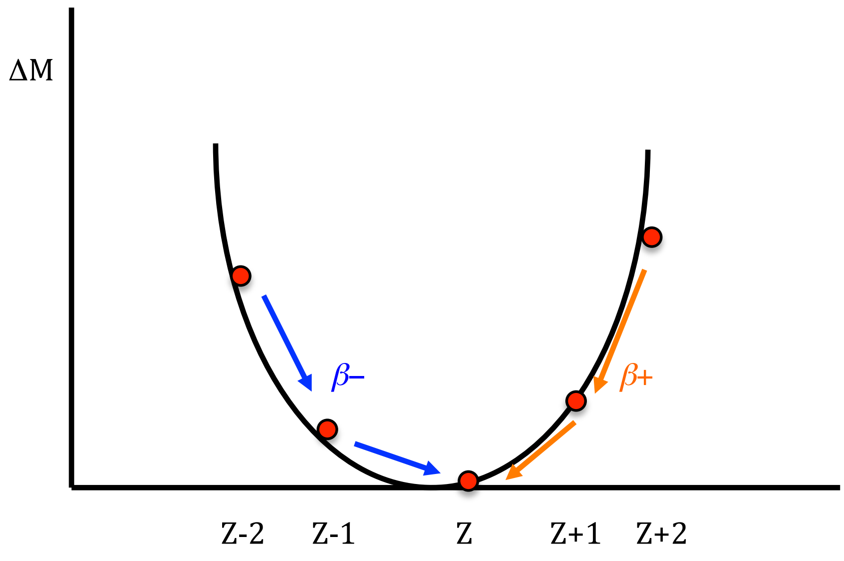
- $\alpha = M_n - a_1 + \frac{a_2}{\mathrm{A}^{1/3}} + \frac{a_4}{4}$
- $\beta = M_n - M_p - M_e + a_4$
- $\gamma = \frac{a_3}{\mathrm{A}^{1/3}} + \frac{a_4}{\mathrm{A}}$
- $\delta = a_5$ (+/- or 0, depending on whether Z & A odd or even).

Some of these parameters are functions of A, but that's OK since we are keeping A constant.

So for any given A we have a quadratic in Z. However, the pairing correction has different effects when A is odd or even.

<u>If A is odd, then for any Z either Z or N must be odd. Thus δ = 0 for all values of Z.</u>

So for odd-A nuclei we obtain a single parabola, which will look something like:



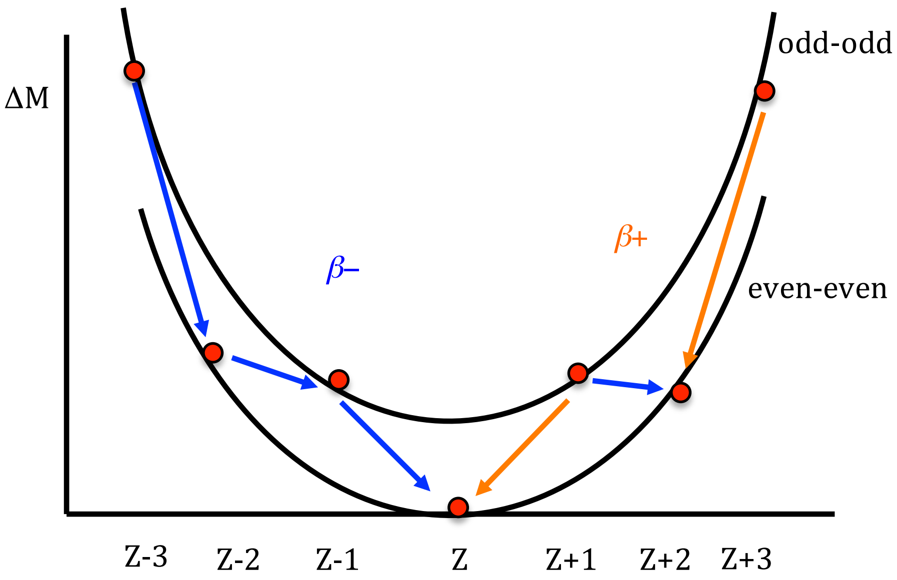
Nuclei on the high-Z side of the minimum may decay via β+ emission or electron capture, while nuclei on the low-Z side decay via β– emission.

If A is even, different Z values will give either even-even or odd-odd nuclei.

Since odd-odd nuclei have a lower binding energy than even-even, we get 2 parabolas for mass vs Z.

The sketch below shows a fairly typical situation:

- The stable isobar is even-even rather than odd-odd. Apart from a few low-mass nuclei, all odd-odd nuclei have one or more more stable even-even isobars.
- There are actually two stable isobars: in this case Z and Z+2. Although the Z+2 isobar is not the most stable one, it is more stable than its immediate neighbours, so cannot decay via either a beta- or a beta+ emission.



In the latter case it may still be possible for the heavier of the 2 stable isobars to decay through the process known as *double beta decay*. This extremely rare process is exactly what it sounds like: 2 neutrons or 2 protons must both decay together, emitting two electrons/positrons and two antineutrinos/neutrinos.

Alpha decay

An alpha particle is a ^{4}He nucleus emitted in the decay of a heavy nucleus. ^{4}He is unusually stable for a very light nucleus, with a binding energy of ~7 MeV/nucleon. The decay is therefore energetically possible is the binding energy of the alpha particle exceeds the difference in binding energies of the initial and final nuclei:

$$B(2,4) > B(Z, A) - B(Z-2, A-4)$$

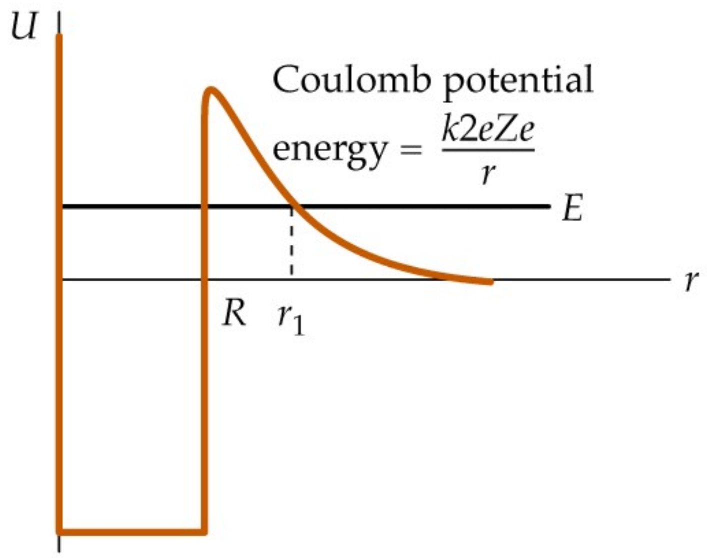
Because B/A decreases gradually for large A, for sufficiently large A all nuclei are unstable to alpha decay. The heaviest stable element is lead (bismuth was thought to be stable until 2003, but is now known to decay via alpha emission with a lifetime of 1.9×10^{19} years).

Alpha decays are known to have a very wide range of lifetimes – from 10ns to 10^{19} years. The lifetime is related to the energy of the emitted alpha by the *Geiger-Nuttall relation*:

$$\log_{10} t_{1/2} = a + b\frac{Z}{E_\alpha^{1/2}}$$

Although this was originally an empirical formula, it is now understood in terms of *quantum tunnelling*. The theory goes approximately like this:

- Nucleons within a nucleus spontaneously cluster to form an alpha particle
- Within the range of the nuclear force, the alpha can be thought of as existing inside a potential well.
- Outside that range, the electrostatic potential of the nucleus forms a potential barrier. If the alpha particle is incident upon the barrier, there is some probability of it "tunnelling" through the barrier.
- The probability of a particle of mass m and energy E tunnelling through a barrier of height V and thickness Δr is $T \approx e^{-2\kappa\Delta r}$, where $\kappa = \left(2m\left|V - E\right|\right)^{1/2}$. We can treat the Coulomb barrier as a series of thin barriers of decreasing height, and so find the total probability of transmission through the barrier



- Combining the probability of transmission per "attempt" with the rate of attempts (rate at which the alpha strikes the barrier) and the probability of the alpha forming in the first place yields a relation similar to the Geiger-Nuttall formula.

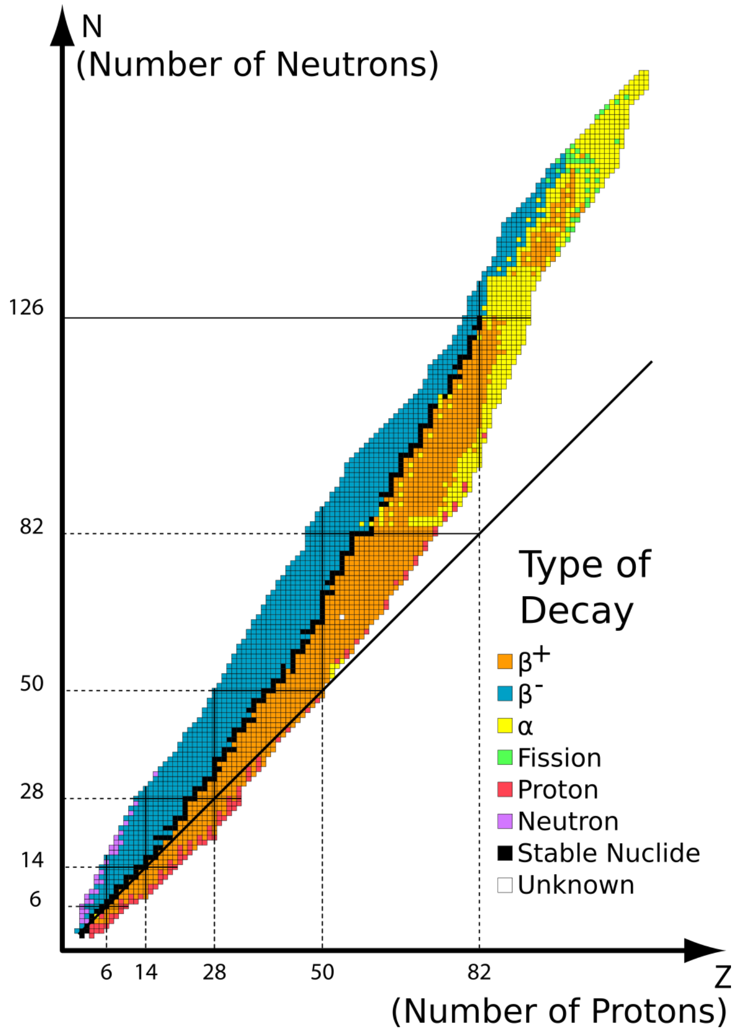
We would now interpret the constant a in the Geiger-Nuttall formula as being related to the probability of finding an alpha in the nucleus, and the second term to the barrier penetration probability and attempt frequency.

Gamma decay

In general, when a nucleus decays through alpha or beta emission, the daughter nucleus is not in its ground state (i.e. the nucleons after the decay are not in their lowest energy states). Therefore it is normal for these decays to be followed by the emission of one or more gamma rays – energetic photons emitted when nucleons change state within the nucleus. The typical lifetime for gamma emission is of the order of 10^{-12}s.

The process is analogous to photon emission due to transitions of atomic electrons between levels. As the spacing between nuclear energy levels is typically larger than between electron shells, nuclear gamma rays are usually more energetic than atomic X-rays, though the energy ranges overlap. Gamma energies are typically of the order of hundreds of keV to a few MeV.

As well as conserving energy in decays, angular momentum must also be conserved. Thus any difference in the spins (angular momenta) of the initial and final nuclei must be carried by the photon. As you have not yet studied the quantum theory of angular momentum I will not discuss the consequences of this here. I will however note that since the photon is a spin 1 particle, one process which is strictly forbidden is the decay of one spin 0 nuclear state into another via gamma radiation.



This chart summarises the stability of nuclei against different types of decay.

The black nuclei form the "line of stability". For low Z this lies close to Z = N, but as Z increases the increasing Coulomb repulsion moves the line of stability to higher N.

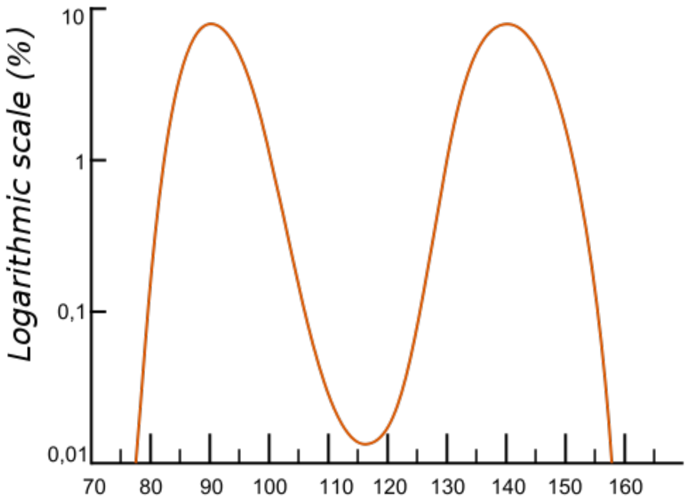
Below the line (in this plot – there are other version) there is an excess of protons, so these species decay through β+ emission. Conversely, above the line where there is a neutron excess β– decay takes place.

At high Z α decays predominate, and in a few cases spontaneous fission can occur (will discuss next lecture).

Finally, when you move very far from the line of stability, nuclei may decay by spontaneous emission of protons or neutrons.

Nuclear Fission

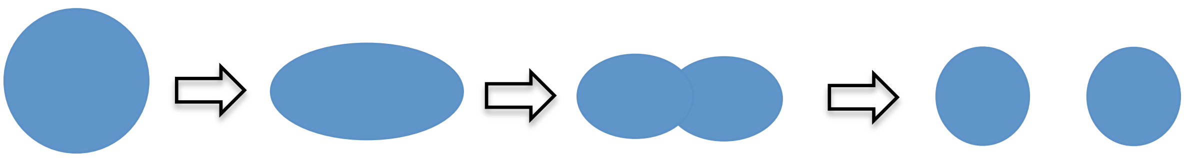
Because B/A declines slowly at high A, it can be energetically favourable for a massive nucleus to split of "fission" into 2 smaller nuclei of approximately equal mass. The maximum energy release should occur when fission is into 2 identical mass fragments, but in practice it is more common for there to be an asymmetry between the masses of the 2 daughter nuclei.



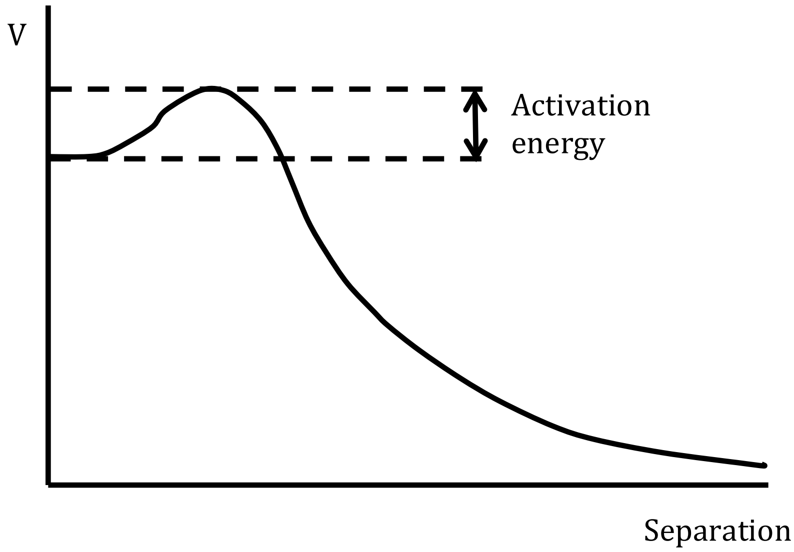
Distribution of Uranium-235 fission products (A)

Since the peak binding energy/nucleon occurs for A ~50-60, *spontaneous fission* (not triggered by an external event) is in principle energetically possible for nuclei with A > ~100. However, in practice spontaneous fission is comparatively rate. To understand why, let us consider the process of nuclear fission.

In the liquid drop model used in the SEMF, the most stable shape for a nucleus is a sphere, since it minimises the surface area for a given volume. Let us imagine that the nucleus is perturbed and becomes elongated. When this happens the "surface energy" is increased (larger surface area/volume, so more nucleons without a full set of neighbours and so less tightly bound) and the Coulomb energy decreased (average distance between protons increases). If the second effect is larger than the first then it will be energetically favourable for the elongation to increase until the nucleus fissions into two smaller, spherical nuclei.



For extremely heavy nuclei this can be the case, in which case any distortion of the shape of the nucleus can lead to fission. More commonly though the initial effect is an increase in potential energy, and only for relatively large distortions does it become energetically favourable for the process to continue (i.e. for the reduction in Coulomb potential to dominate over the reduced nuclear binding). Therefore there is an energy barrier to fission occurring, which must be overcome if fission is to occur.



The difference between the energy of the ground state (spherical) nucleus and the maximum value as it elongates is known as the activation energy. This is typically ~6 MeV for a heavy nucleus. In principle the nucleus could tunnel through this energy barrier, but the tunnelling probability for such massive fragments is very low. However, fission can be triggered if sufficient energy is added to the nucleus, in a process known as *induced fission*.

One way in which fission may be induced is by the capture of a neutron. Because neutrons are not repelled by the charge of the nucleus, they can approach closely enough to be captured by the nuclear force. If this happens the nucleus gains the kinetic energy of the captured neutron, plus a release of binding energy. In addition, if the nucleus has an odd number of neutrons there will be a larger energy release from the pairing energy.

To see the effects, let's compare two neutron capture reactions:

- ${}^{235}_{92}U + n \to {}^{236}_{92}U$ (even-odd → even-even)

  Uranium-236 has an activation energy of ~5 MeV. The binding energy of the last neutron (c^2 x mass difference between ${}^{235}_{92}U + n$ and ${}^{236}_{92}U$) is ~6.5 MeV. Hence the binding energy released in the neutron capture is sufficient to overcome the energy barrier and induce fission in ${}^{236}_{92}U$.

- ${}^{238}_{92}U + n \to {}^{239}_{92}U$ (even-even → even-odd)

  Uranium-239 has an activation energy of ~6.5 MeV. The binding energy of the last neutron is ~4.8 MeV – lower than the last neutron in Uranium-236 since it is unpaired. Here the binding energy released in the neutron capture is not in itself sufficient to induce fission.

Therefore fission of uranium 235 can be triggered even by the capture of a slow (thermal) neutron. However an energetic neutron is required to provide the energy needed to induce fission in uranium 238.

Although we have described fission as the splitting of a single nucleus into two daughters, it is common for one or more neutrons to be released in the reaction as well. It is this which makes the possibility of a *nuclear chain reaction* possible, where neutrons from one fission event induce fission in one or more other nuclei. This is used in nuclear power stations and in nuclear weapons.

The energy released in a fission reaction can be estimated by looking at the diagram of average binding energy/nucleon above. The difference in B/A between a nucleus with A ~ 240 and one with A ~ 120 is approximately 1 MeV/nucleon. Hence the potential energy release in a single nuclear fission could be up to 240 MeV. Normally this is lower, due to the fission into asymmetric fragments (the higher-mass fragment having a smaller difference in B/A than the lower-mass one, as well as a higher A), but is still typically in the range 150-200 MeV. This energy is shared between the kinetic energies of the daughter nuclei and any neutrons released.

For any particular reaction the energy released in the reaction, Q, can be calculated from the difference in masses. So if a nucleus (Z, A) fissions into two daughters, (Z1, A1) and (Z2, A2) plus N neutrons:

$$Q = (M(\mathrm{Z, A}) - M(\mathrm{Z1, A1}) - M(\mathrm{Z2, A2}) - N \times M_n)c^2$$

You can also express this as the difference in binding energies:

$$Q = B(\mathrm{Z, A}) - B(\mathrm{Z1, A1}) - B(\mathrm{Z2, A2})$$

This same principle also applies to the other reactions we have considered previously.

Finally, since higher mass stable nuclei contain proportionately more neutrons than lighter nuclei, the fission of a heavy nucleus will typically produce daughters that are neutron-rich, and hence unstable against $\beta-$ decay.

Nuclear Fusion

While B/A decreases gradually with increasing A for heavy nuclei, it grows very rapidly for light nuclei. Hence if two light nuclei can be *fused* to form a single heavy nucleus the energy released can be very large: fusing two deuterons (B/A = ~1 MeV) to form a ^{3}He nucleus (B/A = ~2.5 MeV) and a neutron releases ~3.5 MeV. While much less than the release in a uranium fission event, the energy released per unit mass is much larger (and this is by no means the most energetic reaction available).

However, for this to happen the two nuclei must overcome their mutual Coulomb repulsion. For any pair of nuclei this can be calculated as:

$$V(r_{sep}) = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 r_{sep}}$$

where Z_1, Z_2 are the charges of the two nuclei and r_{sep} is the separation between their centroids. If for simplicity we assume fusion occurs when $r_{sep} = r_1 + r_2$ (the sum of the nuclear radii) we can estimate the potential barrier which has to be overcome. If we treat the two deuterons of the example above a having radii of ~1fm we would estimate that the Coulomb barrier to deuterium fusion is ~1.4 MeV.

By the standards of nuclear and particle physics this sounds modest – even with heavier nuclei, where the barrier is a few times larger, it would easily be overcome by an accelerator. However, that assumes a true head-on collision – most collisions between pairs of nuclei at those energies would result in elastic scattering rather than fusion. So to produce a high rate of fusion reactions necessary for energy production, whether in a reactor or a stellar interior, requires a dense, hot gas.

Indeed, if we compare that energy barrier to the typical thermal kinetic energy of $1.5*kT$ we find that temperatures of the order of 10^{10} K are needed to achieve this. This is in fact considerably higher than the temperature of a stellar interior (typically 10^8 K). Fusion in stars only occurs because (a) there is a spread in particle energies at a given temperature, with some being much more energetic than the average, and (b) quantum tunnelling allows fusion to occur without the barrier being fully overcome. To discuss these in detail would be beyond the scope of this module.

It's interesting to note that the chain of fusion reactions in the Sun relies on the weak force! Fusing protons (hydrogen) to helium in fact proceeds through three steps:

$$\begin{aligned} {}^1_1H + {}^1_1H &\rightarrow {}^2_1H + e^+ + \nu_e + 0.42\text{ MeV} \\ {}^1_1H + {}^2_1H &\rightarrow {}^3_2He + \gamma + 5.49\text{ MeV} \\ {}^3_2He + {}^3_2He &\rightarrow {}^4_2He + 2\left({}^1_1H\right) + 12.86\text{ MeV} \end{aligned}$$

In each step the energy is released as the kinetic energy of the reaction products. Since the first step is a weak reaction, it proceeds slowly, and this is responsible for the long lifetimes of stars like our Sun.