Magnetic moment of the muon, the search for new fundamental physics and a lattice QCD calculation of hadronic vacuum polarization

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Budapest-Marseille-Wuppertal collaboration [BMWc]
Borsanyi, Fodor, Guenther, Hoelbling, Katz, LL, Lippert, Miura, Szabo, Parato, Stokes, Toth, Torok, Varnhorst

Nature 593 (2021) 51, online 7 April 2021 → BMWc ’20
PRL 121 (2018) 022002 (Editors’ Selection) → BMWc ’17
& Aoyama et al., Phys. Rept. 887 (2020) 1-166 → WP ’20
Lepton magnetic moments and BSM physics
Interaction with an external EM field: QM

Dirac eq. w/ minimal coupling (1928):

\[ i\hbar \frac{\partial \psi}{\partial t} = \left[ \vec{\alpha} \cdot \left( \frac{\hbar}{i} \vec{\nabla} - e\vec{A} \right) + \beta c^2 m + e\ell \vec{A}_0 \right] \psi \]

nonrelativistic limit ↓ (Pauli eq.)

\[ i\hbar \frac{\partial \phi}{\partial t} = \left[ \left( \frac{\hbar}{i} \vec{\nabla} - \frac{e\ell}{c} \vec{A} \right)^2 - \frac{e\ell \hbar}{2m\ell} \vec{\sigma} \cdot \vec{B} + e\ell \vec{A}_0 \right] \phi \]

with

\[ \vec{\mu}_\ell = g_\ell \left( \frac{e\ell}{2m\ell} \right) \vec{S}, \quad \vec{S} = \hbar \frac{\vec{\sigma}}{2} \]

and

\[ g_\ell |_{\text{Dirac}} = 2 \]

“That was really an unexpected bonus for me, completely unexpected.” (P.A.M. Dirac)
Interaction with an external EM field: QFT

Assuming Poincaré invariance and current conservation ($q^\mu J_\mu = 0$ with $q \equiv p' - p$):

$$
\langle \ell(p')|J_\mu(0)|\ell(p)\rangle = \bar{u}(p') \left[ \gamma_\mu F_1(q^2) + \frac{i}{2m_\ell} \sigma_{\mu\nu} q^\nu F_2(q^2) - \gamma_5 \sigma_{\mu\nu} q^\nu F_3(q^2) + \gamma_5 (q^2 \gamma_\mu - 2 m_\ell q_\mu) F_4(q^2) \right] u(p)
$$

- $F_1(q^2) \rightarrow$ Dirac form factor: $F_1(0) = 1$
- $F_2(q^2) \rightarrow$ Pauli form factor, magnetic dipole moment: $F_2(0) = a_\ell = \frac{g_\ell - \frac{2}{2}}{2}$
- $F_3(q^2) \rightarrow \not\!P$, $\not\!T$, electric dipole moment: $F_3(0) = d_\ell / e_\ell$
- $F_4(q^2) \rightarrow \not\!P$, anapole moment: $\not\!\sigma \cdot (\not\!q \times \not\!B)$

- $q^2$ dependence of $F_1(q^2)$ and non-zero $F_2(q^2)$ & $F_{3,4}(q^2)$ come from loops but UV finite once charges and masses are renormalized (in a renormalizable theory)

- $a_\ell$ dimensionless & $m_\gamma = 0$
  - $\Rightarrow$ corrections including only $\ell$ and $\gamma$ are mass independent, i.e. universal
  - $\rightarrow$ contributions from particles w/ $M \gg m_\ell$ are $\propto (m_\ell / M)^{2p} \times \ln^q (m_\ell^2 / M^2)$
  - $\rightarrow$ contributions from pions w/ $m_\pi \ll m_\ell$ are e.g. $\propto \ln(m_\ell^2 / m_\pi^2)$
Why are $a_\ell$ special?

- $a_{e,\mu}$ are parameter-free predictions of the SM that can be measured very precisely $\Rightarrow$ excellent tests of SM

- Loop induced $\Rightarrow$ sensitive to new dofs that may be too heavy or too weakly coupled to be produced directly

- CP and flavor conserving, chirality flipping $\Rightarrow$ complementary to: EDMs, $s$ and $b$ decays, LHC direct searches, . . .

- Chirality flipping $\Rightarrow$ generic contribution of particle w/ $M \gg m_\ell$

  $$a_\ell^M = C \left( \frac{\Delta_{LR}}{m_\ell} \right) \left( \frac{m_\ell}{M} \right)^2$$

- In EW theory, $M = M_W$, chirality flipping from Yukawa, i.e.

  $$\Delta_{LR} = m_\ell \quad \text{and} \quad C \sim \frac{\alpha}{4\pi \sin^2 \theta_W}$$

- In BSM, can have chiral enhancement: e.g. SUSY $M = M_{\text{SUSY}}$ and $C \sim \alpha / (4\pi \sin^2 \theta_W)$ & $\Delta_{LR} = (\mu / M_{\text{SUSY}}) \times \tan \beta \times m_\ell$; or radiative $m_\ell$ model, $\Delta_{LR} \approx m_\ell$, $C \sim 1$ and $M = M_{\Phi}$
Why is $a_\mu$ special?

$m_e : m_\mu : m_\tau = 0.0005 : 0.106 : 1.777 \text{GeV} \quad \tau_e : \tau_\mu : \tau_\tau = \infty : 2 \cdot 10^{-6} : 3 \cdot 10^{-15} \text{s}$

- $a_\mu$ is $(m_\mu/m_e)^2 \sim 4 \times 10^4$ times more sensitive to new $\Phi$ than $a_e$

- $a_\tau$ is even more sensitive to new $\Phi$, but is too shortly lived

- $\tau_\mu$ small but manageable

→ measure & compute $a_\mu$ in SM as precisely as possible

Big question:

$$a_\mu^{\text{exp}} = a_\mu^{\text{SM}}?$$

If not, there must be new $\Phi$
Experimental measurement of $a_\mu$
Measurement principle for $a_\mu$

Precession determined by

$$\vec{\mu}_\mu = 2(1 + a_\mu) \frac{Qe}{2m_\mu} \vec{S}$$

$$\vec{d}_\mu = \eta_\mu \frac{Qe}{2m_\mu c} \vec{S}$$

$$\tilde{\omega}_{a\eta} = \tilde{\omega}_a + \tilde{\omega}_\eta = -\frac{Qe}{m_\mu} \left[ a_\mu \vec{B} + \left( a_\mu - \frac{1}{\gamma^2 - 1} \right) \frac{\vec{\beta} \times \vec{E}}{c} \right] - \eta_\mu \frac{Qe}{2m_\mu} \left[ \frac{\vec{E}}{c} + \vec{\beta} \times \vec{B} \right]$$

- Experiment measures very precisely $\vec{B}$ with $|\vec{B}| \gg |\vec{E}|/c$

  $$\Delta\omega \equiv \omega_S - \omega_C \simeq \sqrt{\omega_a^2 + \omega_\eta^2} \simeq \omega_a$$

  since $d_\mu = 0.1(9) \times 10^{-19} e \cdot cm$ (Benett et al '09)

- Consider either magic $\gamma = 29.3$ (CERN/BNL/Fermilab) or $\vec{E} = 0$ (J-PARC)

  $$\rightarrow \Delta\omega \simeq -a_\mu B \frac{Qe}{m_\mu}$$
$\alpha_\mu = \left( \frac{g_e}{2} \right) \left( \frac{\omega_\alpha}{\omega_p} \right) \left( \frac{\mu_p}{\mu_e} \right) \left( \frac{m_\mu}{m_e} \right)$

$0.26 \text{ ppt}$

$3 \text{ ppb}$

$22 \text{ ppb}$

$2017 \text{ CODATA}$

$N(t) = N_0 e^{-t/t_s} \left[ 1 + A \cos(\omega_a t + \phi) \right]$

$\omega_a$

$\langle \omega_p \rangle \approx \omega_p \otimes \rho(r)$

Average magnetic field weighted by muon distribution

$\omega_p$, free proton precession frequency

Using proton NMR $\hbar \omega_p = 2\mu_p B$

Map the magnetic field

Obtain muon distribution in the storage ring

Esra Barlas-Yucel I FCP 2020

06/10/2020

Laurent Lellouch

Particle Physics Seminar @ U. of Birmingham, 6 October 2021
$g_\mu - 2$ updated history (7 April 2021)

Bathroom scale sensitive to the weight of a single eyelash !!!

Based on only 6% of expected FNAL data! $\rightarrow$ aim $\delta a_\mu = 0.14$ ppm

$\alpha_\mu (\text{AVG}) = 116 \, 592 \, 061(41) \times 10^{-11} \quad (0.35 \text{ ppm})$

G. Venanzoni, CERN Seminar, 8 April 2021
Standard model calculation of $a_\mu$

At needed precision: all three interactions and all SM particles

$$a^{\text{SM}}_\mu = a^{\text{QED}}_\mu + a^{\text{had}}_\mu + a^{\text{EW}}_\mu = O\left(\frac{\alpha}{2\pi}\right) + O\left(\left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_\mu}{M_\rho}\right)^2\right) + O\left(\left(\frac{e}{4\pi \sin \theta_W}\right)^2 \left(\frac{m_\mu}{M_W}\right)^2\right)$$

$$= O\left(10^{-3}\right) + O\left(10^{-7}\right) + O\left(10^{-9}\right)$$
QED contributions to $a_\ell$

Loops with only photons and leptons

$$a_\ell^{\text{QED}} = C_\ell^{(2)} \left( \frac{\alpha}{\pi} \right) + C_\ell^{(4)} \left( \frac{\alpha}{\pi} \right)^2 + C_\ell^{(6)} \left( \frac{\alpha}{\pi} \right)^3 + C_\ell^{(8)} \left( \frac{\alpha}{\pi} \right)^4 + C_\ell^{(10)} \left( \frac{\alpha}{\pi} \right)^5 + \cdots$$

$$C_\ell^{(2n)} = A^{(2n)}_1 + A^{(2n)}_2 (m_\ell / m_{\ell'}) + A^{(2n)}_3 (m_\ell / m_{\ell'}, m_\ell / m_{\ell''})$$

- $A^{(2)}_1$, $A^{(4)}_1$, $A^{(6)}_1$, $A^{(4)}_2$, $A^{(6)}_2$, $A^{(6)}_3$ known analytically (Schwinger ’48; Sommerfield ’57, ’58; Petermann ’57; \ldots)

- $O((\alpha/\pi)^3)$: 72 diagrams (Laporta et al ’91, ’93, ’95, ’96; Kinoshita ’95)

- $O((\alpha/\pi)^4; (\alpha/\pi)^5)$: 891;12,672 diagrams (Laporta ’95; Aguilar et al ’08; Aoyama, Kinoshita, Nio ’96–’18)
  - Automated generation of diagrams
  - Numerical evaluation of loop integrals
  - Not all contributions are fully, independently checked
5-loop QED diagrams

(Aoyama et al '15)
QED contribution to $a_\mu$

\[
\begin{align*}
 a_{\mu}^{\text{QED}} (Cs) &= 1\,165\,847\,189.31(7)m_\tau (17)\alpha^4 (6)\alpha^5 (100)\alpha^6 (23)\alpha (Cs) \times 10^{-12} \quad [0.9 \text{ ppb}] \\
a_{\mu}^{\text{QED}} (a_e) &= 1\,165\,847\,188.42(7)m_\tau (17)\alpha^4 (6)\alpha^5 (100)\alpha^6 (28)\alpha (a_e) \times 10^{-12} \quad [0.9 \text{ ppb}]
\end{align*}
\]

(Aoyama et al '12, '18, '19)

\[
\begin{align*}
 a_{\mu}^{\text{exp}} - a_{\mu}^{\text{QED}} &= 734.2(4.1) \times 10^{-10} \\
 &\equiv a_{\mu}^{\text{EW}} + a_{\mu}^{\text{had}}
\end{align*}
\]
Electroweak contributions to $a_\mu$: $Z$, $W$, $H$, etc. loops

1-loop

$$a^{EW,(1)}_\mu = O\left(\frac{\sqrt{2}G_F m^2_\mu}{16\pi^2}\right)$$

$$= 19.479(1) \times 10^{-10}$$

(Gnendiger et al ’15, Aoyama et al ’20 and refs therein)

2-loop

$$a^{EW,(2)}_\mu = O\left(\frac{\sqrt{2}G_F m^2_\mu}{16\pi^2} \frac{\alpha}{\pi}\right)$$

$$= -4.12(10) \times 10^{-10}$$

(Gnendiger et al ’15 and refs therein)

$$a^{EW}_\mu = 15.36(10) \times 10^{-10}$$
Hadronic contributions to $a_\mu$:

$$a_\mu^{\text{exp}} - a_\mu^{\text{QED}} - a_\mu^{\text{EW}} = 718.9(4.1) \times 10^{-10} \equiv a_\mu^{\text{had}}$$

Clearly right order of magnitude:

$$a_\mu^{\text{had}} = O \left( \left( \frac{\alpha}{\pi} \right)^2 \left( \frac{m_\mu}{M_\rho} \right)^2 \right) = O \left( 10^{-7} \right)$$

(already Gourdin & de Rafael ’69 found $a_\mu^{\text{had}} = 650(50) \times 10^{-10}$)

Write

$$a_\mu^{\text{had}} = a_\mu^{\text{LO-HVP}} + a_\mu^{\text{HO-HVP}} + a_\mu^{\text{HLbyL}} + O \left( \left( \frac{\alpha}{\pi} \right)^4 \right)$$
Hadronic contributions to $a_\mu$: diagrams

$\rightarrow a_{\mu}^{\text{LO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^2\right)$

$\rightarrow a_{\mu}^{\text{NLO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right)$

$\rightarrow a_{\mu}^{\text{HLbL}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right)$
Hadronic light-by-light

- HLbL much more complicated than HVP, but ultimate precision needed is $\sim 10\%$ instead of $\sim 0.2\%$

- For many years, only accessible to models of QCD w/ difficult to estimate systematics (Prades et al '09):
  \[ a_{\mu}^{\text{HLbL}} = 10.5(2.6) \times 10^{-10} \]

- Also, lattice QCD calculations were exploratory and incomplete

- Tremendous progress in past 5 years:
  - Phenomenology: rigorous data driven approach [Colangelo, Hoferichter, Kubis, Procura, Stoffer,…’15-’20]
  - Lattice: first two solid lattice calculations

- All agree w/ older model results but error estimate much more solid and will improve

- Agreed upon average w/ NLO HLbL and conservative error estimates [WP ‘20]

\[ a_{\mu}^{\text{exp}} - a_{\mu}^{\text{QED}} - a_{\mu}^{\text{EW}} - a_{\mu}^{\text{HLbL}} = 709.7(4.5) \times 10^{-10} \Rightarrow a_{\mu}^{\text{HVP}} \]
HVP from $e^+e^- \rightarrow \text{had}$ (or $\tau \rightarrow \nu_\tau + \text{had}$)

Use [Bouchiat et al 61] optical theorem (unitarity)

$$\text{Im}\Pi(s) = -\frac{R(s)}{12\pi}, \quad R(s) \equiv \frac{\sigma(e^+e^- \rightarrow \text{had})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}$$

and a once subtracted dispersion relation (analyticity)

$$\hat{\Pi}(Q^2) = \int_0^\infty ds \frac{Q^2}{s(s+Q^2)} \frac{1}{\pi} \text{Im}\Pi(s)$$

$$= \frac{Q^2}{12\pi^2} \int_0^\infty ds \frac{1}{s(s+Q^2)} R(s)$$

$$\Rightarrow \hat{\Pi}(Q^2) & a_{\mu}^{\text{LO-HVP}} \text{ from data: sum of exclusive } \pi^+\pi^- \text{ etc. channels from CMD-2&3, SND, BES, KLOE '08,'10&'12, BABAR '09, etc.}$$

$$a_{\mu}^{\text{LO-HVP}} = 694.0(1.0)(3.9) \times 10^{-10} [0.6\%]$$

[DHMZ’19] (sys. domin.)

Can also use $I(J^{PC}) = 1(1^-_-)$ part of $\tau \rightarrow \nu_\tau + \text{had}$ and isospin symmetry + corrections
Standard model prediction and comparison to experiment
<table>
<thead>
<tr>
<th>SM contribution</th>
<th>$a_\mu^{\text{contrib.}} \times 10^{10}$</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>HVP LO (R-ratio)</td>
<td>692.8 ± 2.4</td>
<td>[KNT '19]</td>
</tr>
<tr>
<td></td>
<td>694.0 ± 4.0</td>
<td>[DHMZ '19]</td>
</tr>
<tr>
<td></td>
<td>692.3 ± 3.3</td>
<td>[CHHKS '19]</td>
</tr>
<tr>
<td>HVP LO (R-ratio, avg)</td>
<td>693.1 ± 4.0</td>
<td>[WP '20]</td>
</tr>
<tr>
<td>HVP LO (lattice &lt; 2021)</td>
<td>711.6 ± 18.4</td>
<td>[WP '20]</td>
</tr>
<tr>
<td>HVP NLO</td>
<td>−9.83 ± 0.07</td>
<td>[Kurz et al '14, Jegerlehner '16, WP '20]</td>
</tr>
<tr>
<td>HVP NNLO</td>
<td>1.24 ± 0.01</td>
<td>[Kurz '14, Jeger. '16]</td>
</tr>
<tr>
<td>HLbyL LO (pheno)</td>
<td>9.2 ± 1.9</td>
<td>[WP '20]</td>
</tr>
<tr>
<td>HLbyL LO (lattice &lt; 2021)</td>
<td>7.8 ± 3.1 ± 1.8</td>
<td>[RBC '19]</td>
</tr>
<tr>
<td>HLbyL LO (lattice 2021)</td>
<td>10.7 ± 1.1 ± 0.9</td>
<td>[Mainz '21]</td>
</tr>
<tr>
<td>HLbyL LO (avg)</td>
<td>9.0 ± 1.7</td>
<td>[WP '20]</td>
</tr>
<tr>
<td>HLbyL NLO (pheno)</td>
<td>0.2 ± 0.1</td>
<td>[WP '20]</td>
</tr>
<tr>
<td>QED [5 loops]</td>
<td>11658471.8931 ± 0.0104</td>
<td>[Aoyama '19, WP '20]</td>
</tr>
<tr>
<td>EW [2 loops]</td>
<td>15.36 ± 0.10</td>
<td>[Gnendiger '15, WP '20]</td>
</tr>
<tr>
<td>HVP Tot. (R-ratio)</td>
<td>684.5 ± 4.0</td>
<td>[WP '20]</td>
</tr>
<tr>
<td>HLbL Tot.</td>
<td>9.2 ± 1.8</td>
<td>[WP '20]</td>
</tr>
<tr>
<td>SM [0.37 ppm]</td>
<td>11659181.0 ± 4.3</td>
<td>[WP '20]</td>
</tr>
<tr>
<td>Exp [0.35 ppm]</td>
<td>11659206.1 ± 4.1</td>
<td>[BNL '06 + FNAL '21]</td>
</tr>
<tr>
<td>Exp − SM</td>
<td>25.1 ± 5.9 [4.2$\sigma$]</td>
<td></td>
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<td>HVP LO (lattice)</td>
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<td>[BMWc '20]</td>
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<td>10.7 ± 7.0 [1.5σ]</td>
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Very brief introduction to lattice QCD
What is lattice QCD (LQCD)?

To describe matter with sub-% precision, QCD requires $\geq 10^4$ numbers at every spacetime point.

- $\rightarrow \infty$ number of numbers in our continuous spacetime
- $\rightarrow$ must temporarily “simplify” the theory to be able to calculate (regularization)

$\Rightarrow$ Lattice gauge theory $\rightarrow$ mathematically sound definition of NP QCD:

- UV (& IR) cutoff $\rightarrow$ well defined path integral in Euclidean spacetime:

$$
\langle O \rangle = \int \mathcal{D}U \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-S_G - \int \bar{\psi} D[M] \psi} O[U, \psi, \bar{\psi}]
$$

$$
= \int \mathcal{D}U e^{-S_G} \det(D[M]) O[U]_{\text{Wick}}
$$

- $\mathcal{D}U e^{-S_G} \det(D[M]) \geq 0$ & finite # of dofs $\rightarrow$ evaluate numerically using stochastic methods

LQCD is QCD when $m_q \rightarrow m_q^{\text{ph}}, a \rightarrow 0$ (after renormalization), $L \rightarrow \infty$ (and stats $\rightarrow \infty$)

HUGE conceptual and numerical ($O(10^9)$ dofs) challenge
Our “accelerators”

Such computations require some of the world’s most powerful supercomputers

1 year on supercomputer
\[ \sim 100,000 \text{ years on laptop} \]

In Germany, those of the Forschungszentrum Jülich, the Leibniz Supercomputing Centre (Munich), and the High Performance Computing Center (Stuttgart); in France, Turing and Jean Zay at the Institute for Development and Resources in Intensive Scientific Computing (IDRIS) of the CNRS, and Joliot-Curie at the Very Large Computing Centre (TGCC) of the CEA, by way of the French Large-scale Computing Infrastructure (GENCI).
Lattice QCD calculation of $a_{\mu}^{HVP}$

All quantities related to $a_{\mu}$ will be given in units of $10^{-10}$
HVP from LQCD: introduction

Consider in Euclidean spacetime, i.e. spacelike $q^2 = -Q^2 \leq 0$ [Blum '02]

\[
\Pi_{\mu\nu}(Q) = \int d^4x \, e^{iQ \cdot x} \langle J_\mu(x)J_\nu(0) \rangle
\]

\[
= (Q_\mu Q_\nu - \delta_{\mu\nu} Q^2) \, \Pi(Q^2)
\]

w/ $J_\mu = \frac{2}{3} \bar{u}\gamma_\mu u - \frac{1}{3} \bar{d}\gamma_\mu d - \frac{1}{3} \bar{s}\gamma_\mu s + \frac{2}{3} \bar{c}\gamma_\mu c + \cdots$

Then [Lautrup et al '69, Blum '02]

\[
a^{1\text{LO-HVP}} = \alpha^2 \int_0^\infty \frac{dQ^2}{m^2_\ell} \, k(Q^2/m^2_\ell) \hat{\Pi}(Q^2)
\]

w/ $\hat{\Pi}(Q^2) \equiv \left[ \Pi(Q^2) - \Pi(0) \right]$ 

Huge challenge: important long-distance contributions and sub-% precision needed
→ physically light $u$ & $d$ quarks, very large volumes, accurate $Q^2 = 0$ subtraction, many contributions, very high statistics, …

Contributions of $ud, s, c \ldots$ have very different systematics (and statistical errors) on lattice
→ study each one individually

Laurent Lellouch
Particle Physics Seminar @ U. of Birmingham, 6 October 2021
Our lattice definition of $a_{\ell,f}^{\text{LO-HVP}}$

Combining everything, get $a_{\ell,f}^{\text{LO-HVP}}$ from $C_{\text{TL}}^f(t)$ [Bernecker et al ’11, BMWc ’13, Feng et al ’13, Lehner ’14, …]

$$a_{\ell,f}^{\text{LO-HVP}}(Q^2 \leq Q_{\text{max}}^2) = \lim_{a \rightarrow 0, L \rightarrow \infty, T \rightarrow \infty} \alpha^2 \left( \frac{a}{m_\ell^2} \right)^{T/2} \left( \sum_{t=0}^{T/2}' K(tm_\ell, Q_{\text{max}}^2/m_\ell^2) \Re C_{\text{TL}}^f(t) \right)$$

where

$$K(\tau, r_{\text{max}}) = \int_0^{r_{\text{max}}} dr k(r) \left( \tau^2 - \frac{4}{r} \sin^2 \frac{\tau \sqrt{r}}{2} \right)$$

To be competitive w/ R-ratio, had to reduce total uncertainty $19 \times 10^{-10}$ of BMWc ’17 by factor $\approx 4$
Key improvements: statistical noise reduction

Statistical noise of up and down quark contributions increases exponentially with spacetime size of HVP “bubble”

\[ (\alpha/m_\mu)^2 K(t m_\mu) C(t) \times 10^{10} [fm^{-1}] \]

\[ t [fm] \]

(144 \times 96^3, a \sim 0.064 \text{ fm}, M_\pi \sim 135 \text{ MeV})
Key improvements: statistical noise reduction

Statistical noise of up and down quark contributions increases exponentially w/ spacetime size of HVP “bubble”

Solve w/:

- Algorithmic improvements (EigCG, solver truncation [Bali et al ’09], all mode averaging [Blum et al ’13]) to generate more statistics: > 25,000 gauge configurations & tens of millions of measurements

- Exact treatment of long-distance modes to reduce long-distance noise (low mode averaging [Neff et al ’01, Giusti et al ’04, ...])

- Rigorous upper/lower bounds on long-distance contribution [Lehner ’16, BMWc ’17]
### Key improvements: statistical noise reduction

<table>
<thead>
<tr>
<th>Statistical</th>
<th>Error $\times 10^{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.1%</td>
</tr>
<tr>
<td></td>
<td>0.3%</td>
</tr>
<tr>
<td></td>
<td>0.8%</td>
</tr>
<tr>
<td></td>
<td>0.2%</td>
</tr>
<tr>
<td></td>
<td>0.4%</td>
</tr>
<tr>
<td></td>
<td>1.1%</td>
</tr>
<tr>
<td></td>
<td>0.6%</td>
</tr>
<tr>
<td></td>
<td>0.7%</td>
</tr>
<tr>
<td></td>
<td>0.1%</td>
</tr>
</tbody>
</table>

Statistical noise of up and down quark contributions increases exponentially with spacetime size of HVP “bubble”

\[(\alpha / m_\mu)_2 K(t m_\mu) C^{ud}(t) \times 10^{10} [\text{fm}^{-1}]\]

\[t [\text{fm}] \quad da_\mu/dt [\text{BMWc'17}] \quad da_\mu/dt [\text{BMWc'20}]\]

(144 x 96^3, a \sim 0.064 \text{ fm}, M_\pi \sim 135 \text{ MeV})
Must tune parameters of QCD very precisely: $m_u, m_d, m_s, m_c$ & overall mass scale

Solve w/:

- Permil determination of overall QCD scale
- Set w/ $\Omega^-$ baryon mass computed w/ 0.2% uncertainty
- Use Wilson flow scale [Lüscher ’10, BMWc ’12] to separate out electromagnetic corrections
Even on “large” lattices ($L \gtrsim 6 \text{ fm}, T \gtrsim 9 \text{ fm}$), early pen-and-paper estimate [Aubin et al ’16] suggested that exponentially suppressed finite-volume distortions are still $O(2\%)$.

Solve by:

- Finding a way to perform dedicated supercomputer simulations to calculate effect between above and much larger $L = T = 11 \text{ fm}$ volume directly in QCD, i.e. “big” — “ref”

- Computing remnant $\sim 0.1\%$ effect of “big” volume w/ pheno. models of QCD that correctly predict “big” — “ref”
Key improvements: controlled continuum limit

Our world corresponds to spacetime with lattice spacing $a \to 0$. 
Key improvements: controlled continuum limit

Our world corresponds to spacetime with lattice spacing $a \to 0$
Key improvements: controlled continuum limit

Our world corresponds to spacetime w/ lattice spacing $a \rightarrow 0$
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Key improvements: controlled continuum limit

Our world corresponds to spacetime w/ lattice spacing $a \to 0$

Control $a \to 0$ extrapolation of results by:

- Performing all calculations on lattices w/ 6 values of $a$ in range $0.134$ fm $\to 0.064$ fm
- Reducing statistical error at smallest $a$ from 1.9% to 0.3%
- Improving approach to continuum limit w/ pheno. models for QCD [Sakurai '60, Bijnens et al '99, Jegerlehner et al '11, Chakraborty et al '17, BMWc '20] shown to reproduce distortions observed at $a>0$
- Extrapolate results to $a=0$ using theory as guide
Key improvements: QED and $m_u \neq m_d$ corrections

For subpercent accuracy, must include small effects from electromagnetism and due to fact that masses of $u$ and $d$ quarks are not quite equal

- Effects are proportional to powers of $\alpha = \frac{e^2}{4\pi} \sim 0.01$ and $\frac{m_d - m_u}{(M_p/3)} \sim 0.01$

$\Rightarrow$ for SM calculation at permil accuracy sufficient to take into account contributions proportional to only first power of $\alpha$ or $\frac{m_d - m_u}{(M_p/3)}$

- We include all such contributions for all calculated quantities needed in calculation
Thorough and robust determination of statistical and systematic uncertainties

- Stat. err.: resampling methods
- Syst. err.: extended frequentist approach [BMWc '08, '14]
  - Hundreds of thousands of different analyses of correlation functions
  - Weighted by AIC weight
  - Use median of distribution for central values & $16 \div 84\%$ confidence interval to get total error

(Nature paper has 95 pp. Supplementary information detailing methods)
Summary of contributions to $a_{\mu}^{\text{LO-HVP}}$

<table>
<thead>
<tr>
<th>Contribution</th>
<th>Isospin Symmetric</th>
<th>QED Isospin-Breaking:</th>
<th>Strong Isospin-Breaking</th>
<th>Etc.</th>
<th>Finite-Size Effects</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Connected light</td>
<td>Connected valence</td>
<td>Connected disconnected</td>
<td>Bottom; higher order; perturbative</td>
<td>Isospin-symmetric</td>
</tr>
<tr>
<td></td>
<td>633.7(2.1)(4.2)</td>
<td>-1.23(40)(31)</td>
<td>-0.55(15)(10)</td>
<td>0.11(4)</td>
<td>18.7(2.5)</td>
</tr>
<tr>
<td></td>
<td>Connected strange</td>
<td>Connected sea</td>
<td>Disconnected</td>
<td></td>
<td>Isospin-breaking</td>
</tr>
<tr>
<td></td>
<td>53.393(89)(68)</td>
<td>0.37(21)(24)</td>
<td>-0.040(33)(21)</td>
<td></td>
<td>0.0(0.1)</td>
</tr>
<tr>
<td></td>
<td>Connected charm</td>
<td>Disconnected mixed</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>14.6(0)(1)</td>
<td>6.60(63)(53)</td>
<td>-4.67(54)(69)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Disconnected</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-13.36(1.18)(1.36)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$10^{10} \times a_{\mu}^{\text{LO-HVP}} = 707.5(2.3)_{\text{stat}}(5.0)_{\text{sys}}[5.5]_{\text{tot}}$
Comparison and outlook
Consistent with other lattice results

Total uncertainty is divided by $3 \div 4$ . . .

. . . and comparable to R-ratio and experiment

Consistent w/ experiment @ $1.5\sigma$ (“no new physics” scenario) !

$2.1\sigma$ larger than R-ratio average value [WP ’20]
Standard Model with BMWc lattice LO-HVP

Experimental Average

White Paper Standard Model

BNL g-2

FNAL g-2

1.5 $\sigma$

4.2 $\sigma$

$a_\mu \times 10^9 - 1165900$

Laurent Lellouch

Particle Physics Seminar @ U. of Birmingham, 6 October 2021
What next?

- **FNAL** to reduce its error by factor of $\sim 4$ in coming years
- HLbL error must be reduced by factor of $1.5 \div 2$
- Must reduce ours by factor of 4!
- And must reduce proportion of *systematics* in theory error
- Will experiment still agree with our prediction?
- Must be confirmed by other lattice groups
- If confirmed, must understand why lattice doesn’t agree with R-ratio
- If disagreement can be fixed, combine LQCD and phenomenology to improve overall uncertainty [RBC/UKQCD ’18]
- Important to pursue $e^+ e^- \rightarrow \text{hadrons}$ measurements [BaBar, CMD-3, BES III, Belle II, …]
- $\mu e \rightarrow \mu e$ experiment MUonE very important for experimental crosscheck and complementarity w/ LQCD
- Important to build J-PARC $g_\mu - 2$ and pursue $a_\theta$ experiments